

EXACT SOLUTIONS OF THE MORSE-LIKE POTENTIAL, STEP-UP AND STEP-DOWN OPERATORS VIA LAPLACE TRANSFORM APPROACH

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We intend to realize the step-up and step-down operators of the potential $V(x) = V_1 e^{2\beta x} + V_2 e^{\beta x}$. It is found that these operators satisfy the commutation relations for the $SU(2)$ group. We find the eigenfunctions and the eigenvalues of the potential by using the Laplace transform approach to study the Lie algebra satisfied the ladder operators of the potential under consideration. Our results are similar to the ones obtained for the Morse potential ($\beta \rightarrow -\beta$).

1 INTRODUCTION

Searching the exact solutions of the non-relativistic and relativistic wave equations, *i.e.*, Schrödinger equation (SE), Klein-Gordon equation and Dirac equation, has become an important part from the beginning of quantum mechanics [1] and also in the view of the atomic and nuclear physics [2–10].

In this manner, the factorization method by which the creation and annihilation operators of some potentials under consideration could be obtained is a powerful tool to get the exact solutions of some solvable potentials [11, 12]. This approach has been received much attention in order to search the exact solutions in the non-relativistic domain [13]. Bessis and co-workers have extended the Schrödinger-Infeld-Hull factorization method [11, 14] to the case, known as perturbed ladder operator method, where the perturbed eigenvalues and eigenfunctions are written in terms of the unperturbed physical system [15]. Dong and co-workers have also used the factorization method with a different point from the "old" one, namely, the ladder operators under consideration can be constructed in terms of the physical variable (*i.e.*, without using an auxiliary nonphysical variable) to obtain the dynamical group for different types of potentials [16–23].

In this work, our aim is twofold. Firstly, we compute the eigenvalues and the corresponding eigenfunctions of the exponential-type potential, named as the Type-III potential [24], by using the Laplace transform approach (LTA) which is a economic method to obtain the exact solutions of the SE by reducing it into a first-order differential equation [25–27]. Secondly, we intend to search the raising and lowering operators of this potential and give briefly the Lie algebra of the commutators which falls into the $SU(2)$ group. It is seen that our results are similar to the ones obtained for the Morse potential ($\beta \rightarrow -\beta$) [17].

2 ENERGY SPECTRUM

The one dimensional, time-independent Schrödinger equation is written for a particle subjected to the potential under consideration

$$\frac{d^2\phi_n(x)}{dx^2} + \{ME_{n\ell} - MV_1e^{2\beta x} - MV_2e^{\beta x}\}\phi_n(x) = 0, \quad (1)$$

where $M = 2m/\hbar^2$, m is the mass and E is the energy of the particle.

Changing the variable to $z = e^{\beta x}$ in Eq. (1) gives the following equation defined in an interval $z \in [0, \infty]$

$$\frac{d^2\phi_n(z)}{dz^2} + \frac{1}{z} \frac{d\phi_n(z)}{dz} + \left\{ -a_1^2 - \frac{a_2^2}{z} - \frac{\varepsilon^2}{z^2} \right\} \phi_n(z) = 0, \quad (2)$$

where

$$a_1^2 = \frac{MV_1}{\beta^2}; \quad a_2^2 = \frac{MV_2}{\beta^2}; \quad -\varepsilon^2 = \frac{ME_{n\ell}}{\beta^2}. \quad (3)$$

In order to get an equation having a suitable form for applying the Laplace transform approach, we define a wave function $\phi_n(z) = z^\kappa \varphi_n(z)$ which gives

$$z \frac{d^2\varphi_n(z)}{dz^2} - (2\varepsilon + 1) \frac{d\varphi_n(z)}{dz} + \{-a_2^2 - a_1^2 z\} \varphi_n(z) = 0, \quad (4)$$

where we set $\kappa = -\varepsilon$ to obtain a finite wave function when $z \rightarrow \infty$.

By using the Laplace transform defined as [28]

$$\mathcal{L}\{\varphi(z)\} = f(t) = \int_0^\infty dz e^{-tz} \varphi(z), \quad (5)$$

Eq. (4) reads

$$(t^2 - a_1^2) \frac{df(t)}{dt} + [(\varepsilon + 1)t + a_2^2] f(t) = 0, \quad (6)$$

which is a first-order ordinary differential equation and its solution is simply given

$$f(t) \sim (t + a_1)^{-(2\varepsilon+1)} \left(\frac{t - a_1}{t + a_1} \right)^{-\frac{a_2^2}{2a_1} - \frac{2\varepsilon+1}{2}}, \quad (7)$$

In order to obtain a single-valued wave functions, it should be

$$-\frac{a_2^2}{2a_1} - \frac{2\varepsilon + 1}{2} = n \quad (n = 0, 1, 2, \dots) \quad (8)$$

Using this condition and expanding Eq. (7) into series, we obtain

$$f(t) \sim \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)!k!} (2a_1)^k (t+a_1)^{-(2\varepsilon+1)-k}, \quad (9)$$

To obtain the solution of Eq. (4) we use the inverse Laplace transformation [28] and get

$$\varphi_n(z) \sim z^{2\varepsilon} e^{-a_1 z} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)!k!} \frac{\Gamma(2\varepsilon+1)}{\Gamma(2\varepsilon+1+k)} (2a_1 z)^k, \quad (10)$$

which gives

$$\phi_n(z) = N_n z^\varepsilon e^{-a_1 z} \sum_{k=0}^n \frac{(-1)^k n!}{(n-k)!k!} \frac{\Gamma(2\varepsilon+1)}{\Gamma(2\varepsilon+1+k)} (2a_1 z)^k. \quad (11)$$

By using the following definition of the hypergeometric functions [29]

$${}_1F_1(-n, \sigma, \xi) = \sum_{m=0}^n \frac{(-1)^m n!}{(n-m)!m!} \frac{\Gamma(\sigma)}{\Gamma(\sigma+m)} \xi^m, \quad (12)$$

and writing the hypergeometric functions in terms of Laguerre polynomials as $L_n^p(\xi) = \frac{\Gamma(n+p+1)}{n!\Gamma(p+1)} {}_1F_1(-n, p+1, \xi)$ [29], we obtain the eigenfunctions as

$$\phi_n(z) = N_n z^\varepsilon e^{-a_1 z} \frac{n!\Gamma(2\varepsilon+1)}{\Gamma(n+p+1)} L_n^{2\varepsilon}(2a_1 z). \quad (13)$$

Using the normalization condition given as $\int_{-\infty}^{\infty} |\phi(x)|^2 dx = 1$ the normalized eigenfunctions are written

$$\phi_n(z) = \text{const.} \sqrt{\frac{(2\varepsilon)^2 n!}{\Gamma(n+2\varepsilon+1)}} z^\varepsilon e^{-a_1 z} L_n^{2\varepsilon}(2a_1 z), \quad (14)$$

where used [29]

$$\begin{aligned} \int_0^\infty t^{\alpha-1} e^{-\delta t} L_m^\lambda(\delta t) L_n^\beta(\delta t) &= \frac{\delta^{-\alpha} \Gamma(\alpha) \Gamma(n-\alpha+\beta+1) \Gamma(m+\lambda+1)}{m!n! \Gamma(1-\alpha+\beta) \Gamma(1+\lambda)} \\ &\times {}_3F_2(-m, \alpha, \alpha-\beta; -n+\alpha-\beta, \lambda+1; 1) \end{aligned} \quad (15)$$

It is worth to say that 'const.' in Eq. (14) includes some factors related with hypergeometric function $({}_3F_2(a, b, c; r, s; 1))^{-1/2}$ and the parameter a_1 coming from Eq. (15).

The requirement given in Eq. (8) and using the parameters in Eq. (3), we find the energy eigenvalues of the exponential-type potential as

$$E_{n\ell} = -\frac{\beta^2}{4M} \left(2n+1 + \frac{M}{\beta} \frac{V_2}{\sqrt{V_1}} \right)^2. \quad (16)$$

which gives the energy spectra of the Morse potential for $\beta \rightarrow -\beta$. We give numerical energy eigenvalues of the potential obtained from Eq. (16) for two diatomic molecules, namely, for H_2 and LiH molecule. For completeness, we also summarize the eigenvalues for the Morse potential (E_M) by setting $\beta \rightarrow -\beta$ in the same equation. The parameter values of the molecules used here are as follows: $D = 4.7446$ eV, $r_0 = 0.7416$, $m = 0.50391$ amu, $\alpha = \beta r_0 = 1.440558$, $E_0 = \hbar^2/(mr_0^2) = 1.508343932 \times 10^{-2}$ eV for H_2 and $D = 2.515287$ eV, $r_0 = 1.5956$, $m = 0.8801221$ amu, $\alpha = 1.7998368$, $E_0 = 1.865528199 \times 10^{-3}$ eV for LiH molecule [30]. It is seen that being the potential parameter β positive causes to decrease the energy eigenvalues while they increase in the case of the Morse potential.

3 STEP-UP AND STEP-DOWN OPERATORS

In this section, we tend to create briefly the ladder operators of the potential satisfying the following eigenvalue equation

$$\hat{L}_{\pm} \phi_n(z) = \ell_{\pm} \phi_n(z), \quad (17)$$

where \hat{L}_+ is the step-up operator with eigenvalue ℓ_+ and \hat{L}_- is the step-down operator with eigenvalue ℓ_- and having the form [16–23]

$$\hat{L}_{\pm} = f_{\pm}(z) \frac{d}{dz} + g_{\pm}(z). \quad (18)$$

In order to get the step-down operator, we look for the acting of the differential operator d/dz on the eigenfunctions

$$\frac{d}{dz} \phi_n(z) = \frac{\varepsilon}{z} \phi_n(z) - a_1 \phi_n(z) + N_n z^{\varepsilon} e^{-a_1 z} \frac{d}{dz} L_n^{2\varepsilon}(2a_1 z), \quad (19)$$

where if we take into account the constrain $2\varepsilon = -2n - 1 + A$ and use the property of the Laguerre polynomials [31]

$$x \frac{d}{dx} L_n^{\alpha}(x) = n L_n^{\alpha}(x) - (n + \alpha) L_{n-1}^{\alpha}(x), \quad (20)$$

we obtain

$$\left(-z \frac{d}{dz} - a_1 z + n + \varepsilon \right) \phi_n(z) = (n + 2\varepsilon) \frac{N_n}{N_{n-1}} \phi_{n-1}(z), \quad (21)$$

which gives the step-down operator

$$\hat{L}_- = \sqrt{\frac{\varepsilon + 1}{\varepsilon}} \left(-z \frac{d}{dz} - a_1 z + n + \varepsilon \right), \quad (22)$$

with

$$\ell_- = (-n + A - 1) \sqrt{n(n + 2\varepsilon + 1)}. \quad (23)$$

where $A = -a_2^2/a_1$. The last equation shows that the step-down operator destroys the ground state. Using the following recursion relation of the Laguerre polynomials [31]

$$x \frac{d}{dx} L_n^\alpha(x) = (n+1) L_{n+1}^\alpha(x) - (n+\alpha+1-x) L_n^\alpha(x), \quad (24)$$

and inserting into Eq. (19) gives

$$\left(z \frac{d}{dz} - z + a_1 z + n + \varepsilon + 1 \right) \phi_n(z) = (n+1) \frac{N_n}{N_{n+1}} \phi_{n+1}(z), \quad (25)$$

From the last equation we obtain the step-up operator as

$$\hat{L}_+ = \sqrt{\frac{\varepsilon-1}{\varepsilon}} \left(z \frac{d}{dz} - z + a_1 z + n + \varepsilon + 1 \right), \quad (26)$$

with

$$\ell_+ = \sqrt{\frac{n+1}{-n+A+1}}. \quad (27)$$

The step-up operator in Eq. (26) annihilates the last bounded state since for a such state is $\varepsilon = 1$.

Finally we study the Lie algebra associated to the operators \hat{L}_\pm to construct the commutator of them with the help of Eqs. (22) and (26):

$$[\hat{L}_+, \hat{L}_-] \phi_n(z) = \ell_0 \phi(z) \quad (28)$$

where the eigenvalue

$$\ell_0 = 2n + 2 - A, \quad (29)$$

which makes it possible to construct the operator

$$\hat{L}_0 = 2\hat{n} + 2 - A. \quad (30)$$

These three operators satisfy the following Lie algebra

$$[\hat{L}_+, \hat{L}_-] = \hat{L}_0; [\hat{L}_-, \hat{L}_0] = \hat{L}_-; [\hat{L}_0, \hat{L}_+] = \hat{L}_+. \quad (31)$$

which correspond to the SU(2) group of the potential that means the potential under consideration has the same group of the Morse potential [17].

4 CONCLUSION

We have obtained the ladder operators of the Type-III potential and commutation relations which correspond to the SU(2) group which is also correspond to the ones of the Morse potential. To achieve this aim, we have solved the Schrödinger equation for the potential under consideration by using the Laplace transform approach to find the eigenfunctions and eigenvalues. We have also obtained the energy values of the Morse potential by setting $\beta \rightarrow -\beta$ and summarized our numerical results obtained for two diatomic molecules in Table 1.

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Table 1: Energy eigenvalues of the exponential-type and Morse potentials for different values of n in eV ($V_1 = D, V_2 = 2D$).

n	H_2			LiH		
	$E_{n\ell} < 0$	$E_M < 0$	Ref. [30]	$E_{n\ell} < 0$	$E_M < 0$	Ref. [30]
0	5.02101	4.47601	4.47601	2.60322	2.42886	2.42886
2	6.20491	3.47992	3.47991	2.97007	2.09828	2.09827
4	7.51402	2.60903	2.60902	3.36109	1.79186	1.79186
10	12.1926	0.74759	0.74759	4.67918	1.01766	1.01765